PHYSICAL REVIEW VOLUME 135, NUMBER 2B 27 JULY 1964

Mixing Effects for ϕ, ω, and ρ^0 Mesons*

JOHN HARTE AND ROBERT G. SACHS†
University of Wisconsin, Madison, Wisconsin
(Received 10 February 1964)

A method based on the properties of the propagator which was developed for treating the time-dependent behavior of almost degenerate unstable particles is used to discuss mixing effects in resonance phenomena. General equations for the location of the poles and the mixing of states in the production amplitudes are presented. The method is applied to the mixing of ϕ, ω, and ρ^0 mesons. Mixing coefficients and poles are determined explicitly in terms of the elements of the matrix representing the square of the mass. In particular an expression is given for the complex amplitude, s, of ϕ, ω mixing but it is indicated that $\mathrm{Im}s \ll \mathrm{Re}s$ so that the usual approximation of treating s as real is probably good enough for most purposes. The 2π and 3π production amplitudes due to ρ^0 and ω production are shown to depend strongly on the production mechanism, as already noted by Bernstein and Feinberg. For example it is found that production of the 2π mode will have its maximum at the ω pole in a process dominated by ρ exchange. It is also shown that ρ^0 and ω masses could be taken to be nearly equal if the apparent splitting of the masses is caused by destructive interference. It is found that, in general, PT invariance implies that the mass matrix is symmetric.

I. INTRODUCTION

THE recently discovered existence of a number of quasistable, strongly interacting particles has stimulated an interest in mixing effects between such systems,[1] particularly the mixing of pairs of particles having nearly the same mass. A discussion of the mixing due to electromagnetic interactions, with particular reference to the mixing of ρ^0 and ω particles, has been presented by Bernstein and Feinberg.[2] Their method was based on the Weisskopf-Wigner perturbation theory. A more general method[3] is provided by considering the properties of the propagator matrix of the system of almost degenerate particles. The purpose of this paper is to elucidate this more general approach and to apply it to several specific problems.

Those properties of the particle needed to describe its contribution to a scattering matrix are assumed to be completely expressed by a propagator. This should be a good approximation for a quasistable particle, that is, if the lifetime is long enough. The propagator appears in the scattering amplitude as a factor sandwiched between vertex functions referring to the particular processes in which the unstable particle is produced and detected. The vertex functions are assumed to be slowly varying functions of energy over a range corresponding to a reciprocal lifetime of the particle (that is, over the width of the resonance).

In order to establish certain general symmetry properties of the propagator, it is convenient to assume that, in a space-time representation, it may be formally written as a vacuum expectation value of a time-ordered product of local, Heisenberg field operators. The consequences of P, C, and T invariance are easily obtained in this way.[4] In particular, it will be shown later that PT invariance guarantees that the complex mass matrix of the coupled particles is symmetric, a point that was left unsettled by Bernstein and Feinberg.[2]

II. ANALYSIS OF THE PROPAGATOR MATRIX

In order to give our notation some physical content, we may consider the specific problem of ω, ϕ, ρ^0 mixing, although it should be clear that the methods are general. We start from a set of "bare" states which are denoted by latin indices. These states are then denoted by $|i\rangle$ where, for the above mentioned case, $i=1$ will refer to the $I=0$ singlet under SU_3, while $i=2$, 3 refer to the $I=0$ and $I=1$ members of the vector meson octet. Thus, in defining the bare states it is presumed that both the symmetry breaking and electromagnetic interactions are turned off. Since the latter interaction is much weaker than the former, these interactions may be turned on in two steps, However, for the purpose of the general argument, it is assumed that this is accomplished in one step, thereby producing the complete physical states denoted by a greek index $\alpha=\rho^0$, ω, or ϕ.

The propagator is assumed to be given in a representation referring to the bare states since it is in this way that the role of the various symmetry breaking interactions can be assessed. It is then a 3×3 matrix, $\langle i|\Delta_F'(k^2)|j\rangle$ where k is the four-momentum of the particles and $k^2>0$ is taken to be time-like.

* Work supported in part by the University of Wisconsin Research Committee with funds granted by the Wisconsin Alumni Research Foundation and in part by the U. S. Atomic Energy Commission under Contract No. AT(11-1)-30.

† New address: High Energy Physics, Argonne National Laboratory, Argonne, Illinois.

[1] S. Glashow, Phys. Rev. Letters **7**, 469 (1961).

[2] J. Bernstein and G. Feinberg, Nuovo Cimento **25**, 1343 (1962).

[3] R. Jacob and R. G. Sachs, Phys. Rev. **121**, 350 (1961). S. Coleman and H. Schnitzer have also recently noted that the mixing effects could be discussed in terms of the propagator (unpublished). In particular they show that the symmetry of the mass matrix may be obtained from CPT invariance, which is essentially the same as our argument based on PT invariance for these neutral particles. See also G. Feldman and P. Matthews, Phys. Rev. **132**, 823 (1963).

[4] The consequences of CP and CPT invariance for the neutral K mesons have been obtained in this way. R. G. Sachs, Ann. Phys. **22**, 239 (1963). The present paper is an extension of the same methods to situations in which the particle is observed as a resonance in a reaction process, whereas the earlier discussion was concerned with explicitly time-dependent processes such as particle decay.

The matrix $\Delta_F'(k^2)$ is given as the boundary value[5]

$$\Delta_F'(k^2)=\lim_{z\to k^2+i\epsilon} F(z) \tag{1}$$

of a 3×3 matrix function $F(z)$ which is analytic in the z plane with a cut on the positive real axis. It is convenient to write

$$F(z)=[z-W(z)]^{-1}, \tag{2}$$

where $W(z)$ is the 3×3 matrix consisting of the sum of the bare mass and all the proper self-energy contributions, including those due to electromagnetic and SU_3 symmetry breaking effects to the desired order.

The physical particles are characterized by the poles in $F(z)$ which are nearest the physical sheet. If Eq. (2) is taken to be the definition of $W(z)$ on unphysical as well as physical sheets, the location of such a pole z_α is determined by the solutions of the equation

$$\det[z_\alpha-W(z_\alpha)]=0. \tag{3}$$

The solutions of interest are those nearest the physical sheet ($z=k^2+i\epsilon$) since the pole then corresponds to a decaying particle.[3] It is expected that in the n-particle problem there will be just n of these poles near the physical region. Henceforth only those n physical poles are considered. If the particle is observed as a resonance in a reaction process, then the location of the resonance is determined by $\mathrm{Re}(z_\alpha)$ and the width by $\mathrm{Im}(z_\alpha)$.

According to Eq. (3), we may solve for the left and right eigenfunctions of W, which are determined by

$$\begin{aligned}&[z_\alpha-W(z_\alpha)]\psi_\alpha=0,\\ &\psi_\alpha^\dagger[z_\alpha-W(z_\alpha)]=0.\end{aligned} \tag{4}$$

Let us now assume that in the vicinity of the pole z_α, $F(z)$ has the form

$$F(z)=\Phi_\alpha(z-z_\alpha)^{-1}+Q_\alpha(z), \tag{5}$$

where Φ_α is a constant matrix and Q_α is regular at $z=z_\alpha$. Then Eq. (2) implies

$$(z-z_\alpha)^{-1}[z-W(z)]\Phi_\alpha+[z-W(z)]Q_\alpha(z)=1$$

and

$$(z-z_\alpha)^{-1}\Phi_\alpha[z-W(z)]+Q_\alpha(z)[z-W(z)]=1.$$

For these equations to be valid at the singular point $z=z_\alpha$, we must have $\Phi_\alpha=\psi_\alpha\otimes\psi_\alpha^\dagger$. It follows that

$$F(z)=\sum_{\alpha=1}^{n}\frac{\psi_\alpha\otimes\psi_\alpha^\dagger}{z-z_\alpha}+Q(z), \tag{6}$$

where $Q(z)$ is well-behaved in the vicinity of the physical poles z_α.

It will be shown in the next section that the matrix $W(z)$ is symmetric if all interactions are PT invariant. Therefore, for any given z, $W(z)$ may be brought to diagonal form by a complex orthogonal transformation $\Omega(z)$. If we denote the columns of $\Omega(z)$ by $\Omega_\alpha(z)$, then they are determined by the eigenvalue problem

$$W(z)\Omega_\alpha(z)=\lambda_\alpha(z)\Omega_\alpha(z). \tag{7}$$

The $\lambda_\alpha(z)$ are given by the secular equation

$$\det[W(z)-\lambda_\alpha(z)]=0. \tag{8}$$

From Eq. (3) it is clear that the poles z_α are determined by the equations

$$z_\alpha=\lambda_\alpha(z_\alpha), \tag{9}$$

where the ordering of the labels is appropriately prescribed. Furthermore, from Eq. (4),

$$\psi_\alpha=\Omega_\alpha(z_\alpha) \tag{10}$$

and

$$\psi_\alpha^\dagger=\tilde{\Omega}_\alpha(z_\alpha)=\tilde{\psi}_\alpha, \tag{11}$$

where the tilda denotes a row vector (the transpose of ψ_α).

In the case that $W(z)$ may be treated as a constant, independent of z, over the entire range of interest, the transformation Ω is independent of z and the ψ_α are simply the columns of Ω. Therefore, $\psi_\alpha^\dagger$ and ψ_α form an orthonormal set of vectors under this condition. Furthermore, as a consequence of trace invariance

$$\textstyle\sum_\alpha z_\alpha=\sum_\alpha \lambda_\alpha=\mathrm{tr}W \tag{12}$$

if W is constant. Note that Eq. (12) does not apply in general if W is a function of z since then

$$\textstyle\sum_\alpha z_\alpha=\sum_\alpha \lambda_\alpha(z_\alpha)\neq\mathrm{tr}W(z). \tag{13}$$

III. SYMMETRY OF THE PROPAGATOR MATRIX

In order to determine the symmetry properties of the propagator, it is convenient to assume that it can be represented in configuration space in terms of certain local field operators denoted by $\varphi_i(x)$. The assumption is that $\varphi_i(x)$ is a Heisenberg field operator which in the asymptotic limit, $t\to-\infty$, creates or annihilates a one particle state having exactly the quantum numbers denoted by the label i. Then the propagator matrix takes the form[6]

$$\begin{aligned}\langle i|\Delta_F'(x)|j\rangle&=\langle 0|\{\varphi_i(\tfrac{1}{2}x)\varphi_j(-\tfrac{1}{2}x)\}_T|0\rangle\\ &=\theta(x)\langle 0|\varphi_i(\tfrac{1}{2}x)\varphi_j(-\tfrac{1}{2}x)|0\rangle\\ &\quad+\theta(-x)\langle 0|\varphi_j(-\tfrac{1}{2}x)\varphi_i(\tfrac{1}{2}x)|0\rangle,\end{aligned} \tag{14}$$

where $|0\rangle$ is the vacuum state, $\{\ \}_T$ denotes the time ordered product and $\theta(x)$ is the usual step function in time.

From the form of Eq. (14), it can be shown that the propagator matrix is symmetric if all interactions are invariant under PT. In that case the Heisenberg fields

[5] The formal treatment here is presented for scalar fields. For the vector fields associated with spin-1 particles such as the ρ, ω, ϕ mesons, the propagator contains a factor of the form $\delta_{\mu\nu}-k_\mu k_\nu/m^2$ which is not expressly included here and does not affect the points under discussion.

[6] For a formal demonstration of this, see the Appendix.

satisfy the condition

$$PT\varphi_i(x)(PT)^{-1}=\varphi_i(-x). \quad (15)$$

Since PT is an anti-unitary operator we may write

$$\begin{aligned}\langle 0|\,\varphi_i(\tfrac{1}{2}x)\varphi_j(-\tfrac{1}{2}x)\,|0\rangle \\ =\langle PT(0)\,|\,PT\varphi_i(\tfrac{1}{2}x)\varphi_j(-\tfrac{1}{2}x)\,|0\rangle^*,\end{aligned} \quad (16)$$

where $|PT(0)\rangle=|0\rangle$ is the PT reversed vacuum state. Therefore,

$$\begin{aligned}&\langle 0|\,\varphi_i(\tfrac{1}{2}x)\varphi_j(-\tfrac{1}{2}x)\,|0\rangle \\ &=\langle 0|\,PT\varphi_i(\tfrac{1}{2}x)(PT)^{-1}(PT)\varphi_j(-\tfrac{1}{2}x)(PT)^{-1}|0\rangle^* \\ &=\langle 0|\,\varphi_i(-\tfrac{1}{2}x)\varphi_j(\tfrac{1}{2}x)\,|0\rangle^*\end{aligned} \quad (17)$$

by Eq. (15). Thus

$$\langle 0|\,\varphi_i(\tfrac{1}{2}x)\varphi_j(-\tfrac{1}{2}x)\,|0\rangle=\langle 0|\,\varphi_j(\tfrac{1}{2}x)\varphi_i(-\tfrac{1}{2}x)\,|0\rangle. \quad (18)$$

A similar result applies to the second term in Eq. (14) whence it follows that

$$\langle i|\,\Delta_F'(x)\,|j\rangle=\langle j|\,\Delta_F'(x)\,|i\rangle. \quad (19)$$

The Fourier transform $\Delta_F'(k^2)$ of the matrix $\Delta_F'(x)$ is therefore also symmetric and we may take its analytic continuation $F(z)$ to be a symmetric matrix:

$$\langle i|F(z)\,|j\rangle=\langle j|F(z)\,|i\rangle. \quad (20)$$

Then since the inverse of a symmetric matrix is symmetric,

$$\langle i|\,W(z)\,|j\rangle=\langle j|\,W(z)\,|i\rangle, \quad (21)$$

where $W(z)$ is defined by Eq. (2).

IV. PRODUCTION AMPLITUDES

Under the assumptions stated in Sec. I, the production amplitude will have the form indicated by the typical production diagram shown in Fig. 1, that is, the amplitude is a product of a production vertex indicated by the shaded box in the diagram, a propagator $\langle j|\Delta_F'|i\rangle$ indicated by the heavy line ij, and a decay vertex D_j. The lines labeled a, b, c, $\cdots$ are meant to symbolize an arbitrary set of incoming and outgoing particles.

D_j and V_i are assumed to be independent of k^2 over the energy range in which the pole terms in the propagator dominate. Then the amplitude for production of the final state d, for some production process p, which proceeds via the intermediate particle states $|\alpha\rangle$, is given, for fixed momentum transfer, by

$$A_{dp}(k^2)=\sum_{i,j}D_j(d)\langle j|\,\Delta_F'(k^2)\,|i\rangle V_i(p). \quad (22)$$

The factor $\langle j|\Delta_F'(k^2)|i\rangle$ can be rewritten in a more useful form in terms of the elements of the matrix W, by taking matrix elements of Eq. (6). Thus, according to Eqs. (1), (6), and (11)

$$\langle j|\,\Delta_F'(k^2)\,|i\rangle=\sum_\alpha \psi_{j\alpha}\psi_{i\alpha}/[k^2-z_\alpha]+Q_{ji}(k^2), \quad (23)$$

where

$$\psi_{i\alpha}=\langle i|\psi_\alpha\rangle.$$

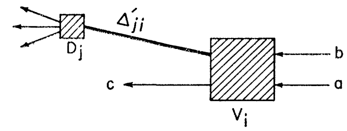

FIG. 1. Typical diagram for production and decay of unstable particle. The boxes labeled V_i and D_j designate arbitrary production and decay processes for the unstable particle states denoted by i, j. Incoming and outgoing particles are designated by a, b, c, $\cdots$.

The amplitude may then be written in the form

$$A_{dp}(k^2)=\sum_\alpha [D_\alpha(d)V_\alpha(p)/(k^2-z_\alpha)]+R_{dp}(k^2), \quad (24)$$

where

$$V_\alpha=\sum_i \psi_{i\alpha}V_i \quad (25a)$$

and

$$D_\alpha=\sum_i D_i\psi_{i\alpha} \quad (25b)$$

are the production and decay vertices for particles of type α, while

$$R_{dp}(k^2)=\sum_{i,j}D_j(d)Q_{ji}(k^2)V_i(p)$$

is a slowly varying remainder function.

This amplitude describes a process in which several resonances are participating. Since z_α corresponds to the square of the complex mass, it is convenient to write

$$z_\alpha=(M_\alpha-\tfrac{1}{2}i\Gamma_\alpha)^2, \quad (26)$$

where M_α is the real mass and Γ_α is the width of the resonance. Since Γ_α is usually much less than M_α, it is a good approximation to write

$$\begin{aligned}M_\alpha^2&\approx \mathrm{Re}z_\alpha, \\ \Gamma_\alpha&\approx -M_\alpha^{-1}\,\mathrm{Im}z_\alpha.\end{aligned} \quad (27)$$

V. STRUCTURE OF THE ω, ρ^0, ϕ PROPAGATOR

We shall now apply these general considerations to the specific problems of ω, ρ^0, ϕ mixing. In accordance with the notation suggested in Sec. 2, the "bare state" labels $i=1$, 2, 3 are defined as follows:

$i=1$	unitary singlet	$I=0$,
$i=2$	unitary octet	$I=0$,
$i=3$	unitary octet	$I=1$,

where the charge is zero and $S=0$ for all three states. Thus $i=3$ corresponds to the "bare" ρ^0 particle, while $i=1$ and 2 refer to the "bare" ω and ϕ particles. Then in the $|i\rangle$ representation the matrix W takes the form

$$W(z)=\begin{bmatrix} W_1(z) & t(z) & e(z) \\ t(z) & W_2(z) & e'(z) \\ e(z) & e'(z) & W_3(z)\end{bmatrix}. \quad (28)$$

The z dependence of each of the matrix elements may be determined from the fact that $W(z)$ is expected to have the form of a complex (mass)^2 matrix

$$W(z)=R(z)-i\sum_c \beta_c(z)(z-b_c)^{\frac{1}{2}}, \quad (29)$$

where R and β_c are slowly varying matrix functions of z, b_c is the threshold for channel c, and the $\frac{3}{2}$ power appears because at the threshold for each channel the outgoing particles are in p states.

We may define a complex mass $m_j = M_j - \frac{1}{2}i\Gamma_j$ for the "bare" state denoted by $|j\rangle$ as the solution of the equation

$$W_j(m_j^2) = m_j^2, \tag{30}$$

which would follow from Eq. (3) if there were no mixing. The mixing of the singlet and octet states is governed by the symmetry breaking coupling t, which is almost of the same order of magnitude as the W_j. On the other hand, e and e' are electromagnetic couplings which are expected to be of the order of $t/137$.

Solution of the eigenvalue problem Eqs. (3) and (4) for this case is more transparent from a physical point of view if it is carried out in two steps: first the submatrix

$$w(z) = \begin{pmatrix} W_1(z) & t(z) \\ t(z) & W_2(z) \end{pmatrix}$$

is diagonalized and then W is rewritten in the new basis and the diagonalization is completed. The first step serves to define states $|\phi^{(0)}\rangle$ and $|\omega^{(0)}\rangle$ corresponding to the "physical" ϕ and ω particles in the absence of the electromagnetic mixings. The final step produces the true physical states $|\phi\rangle$ and $|\omega\rangle$.

VI. DETERMINATION OF $\omega^{(0)}$ AND $\phi^{(0)}$ STATES

The complex masses of the particles denoted by $\phi^{(0)}$ and $\omega^{(0)}$ are determined from the roots $z_\alpha^{(0)}$ of the secular equation

$$\det[w(z_\alpha^{(0)}) - z_\alpha^{(0)}] = 0, \tag{31}$$

which are

$$z_\phi^{(0)} = \tfrac{1}{2}[W_1(z_\phi^{(0)}) + W_2(z_\phi^{(0)}) + r(z_\phi^{(0)})], \tag{32}$$

$$z_\omega^{(0)} = \tfrac{1}{2}[W_1(z_\omega^{(0)}) + W_2(z_\omega^{(0)}) - r(z_\omega^{(0)})], \tag{33}$$

where we have introduced

$$r(z) = \{[W_2(z) - W_1(z)]^2 + 4t^2(z)\}^{\frac{1}{2}}.$$

The solutions of the eigenvector equations

$$w(z_\alpha^{(0)})\psi_\alpha^{(0)} = z_\alpha^{(0)}\psi_\alpha^{(0)} \tag{34}$$

are given by the transformations

$$\begin{aligned} \psi_\omega^{(0)} &= (1+s_\omega^2)^{-1/2}[|1\rangle + s_\omega|2\rangle], \\ \psi_\phi^{(0)} &= (1+s_\phi^2)^{-1/2}[-s_\varphi|1\rangle + |2\rangle], \end{aligned} \tag{35}$$

where

$$s_\alpha = [W_2(z_\alpha^{(0)}) - W_1(z_\alpha^{(0)}) - r(z_\alpha^{(0)})]/2t(z_\alpha^{(0)}). \tag{36}$$

It is useful to note that

$$s_\alpha^2 = [z_\alpha^{(0)} - W_2(z_\alpha^{(0)})]/[z_\alpha^{(0)} - W_1(z_\alpha^{(0)})] \tag{37}$$

and also

$$\begin{aligned} z_\omega^{(0)} &= W_1(z_\omega^{(0)}) + t(z_\omega^{(0)})s_\omega \\ z_\phi^{(0)} &= W_2(z_\phi^{(0)}) - t(z_\phi^{(0)})s_\phi. \end{aligned} \tag{38}$$

Although from Eq. (29) it is clear that $W(z)$ is sensitive to z near a threshold, only the imaginary part of $W(z)$ is significantly affected since the widths, Γ_α, are small compared to the masses M_α. Therefore we may take $\mathrm{Re}(W(z))$ to be independent of z. Furthermore, we shall show later that s_α^2, as given by Eq. (37), is insensitive to the values of $\mathrm{Im}W_j(z)$ within a reasonable range. Therefore, for all practical purposes, the z dependence of W may be ignored in evaluating Eq. (37) and $W_i(z)$ may be replaced by m_i^2, which is given by Eq. (30). Then

$$s_\omega^2 = s_\phi^2 = s^2 = (z_\phi^{(0)} - m_2^2)/(z_\phi^{(0)} - m_1^2). \tag{39}$$

Again, since $t(z_\omega^{(0)}) - t(z_\phi^{(0)})$ is of the order of Γ_α, the sum of Eqs. (38) leads to

$$z_\phi^{(0)} + z_\omega^{(0)} = m_1^2 + m_2^2 \tag{40}$$

which agrees with Eq. (12).

By means of Eqs. (39) and (40) it is possible to determine s^2 from a knowledge of the complex physical masses (assuming electromagnetic effects are small) and a presumption about the complex "bare" mass m_2. For example, recent experimental evidence[7] indicates that

$$\begin{aligned} M_\phi &\approx 1.018 \text{ BeV}, \\ \Gamma_\phi &\approx 0.003 \text{ BeV}, \\ M_\omega &\approx 0.784 \text{ BeV}, \\ \Gamma_\omega &\approx 0.010 \text{ BeV}. \end{aligned} \tag{41}$$

From these values, $z_\phi^{(0)}$ and $z_\omega^{(0)}$ may be obtained by means of Eq. (27) since the difference between z_α and $z_\alpha^{(0)}$ for $\alpha = \omega, \phi$ is presumably small. If, in addition, we assume the Okubo mass formula to hold with the bare ϕ particle chosen to be a member of the vector meson octet, then the mass formula

$$\mathrm{Re}(m_{K^*}^2) = \tfrac{1}{4}\,\mathrm{Re}(m_\rho^2) + \tfrac{3}{4}\,\mathrm{Re}(m_2^2) \tag{42}$$

provides the value $\mathrm{Re}(m_2^2) = (0.930 \text{ BeV})^2$. Im m_2^2 cannot be obtained by such direct means, but, as mentioned earlier, s^2 turns out to be fairly insensitive to its value. This can be seen from the following tabulation of s^2 for a wide variety of values of Γ_2:

$$\begin{aligned} &\Gamma_2 = 0.03 \text{ BeV}, && s^2 = 0.64 + 0.16i, && s = 0.8 + 0.1i, \\ &\Gamma_2 = 0.01 \text{ BeV}, && s^2 = 0.64 + 0.03i, && s = 0.8 + 0.02i, \\ &\Gamma_2 = 0.001 \text{ BeV}, && s^2 = 0.64 - 0.03i, && s = 0.8 - 0.02i. \end{aligned} \tag{43}$$

We note that the connection between s and the mixing parameter $\sin\lambda$ introduced by Sakurai[8] is

$$\sin\lambda = -s(1+s^2)^{-\frac{1}{2}} \tag{44}$$

in the approximation that the mass matrix, and therefore s, is real. This approximation leads to $s \approx 0.8$

[7] N. Gelfand, D. Miller, M. Nussbaum, J. Ratau, J. Schultz *et al.*, Phys. Rev. Letters **11**, 436, 438 (1963).

[8] J. J. Sakurai, Phys. Rev. Letters **9**, 472 (1962).

because all the widths involved in the mass matrix are small. The value $s=0.8$ leads to $\lambda\approx 39°$ in agreement with the value obtained by Sakurai.

In the approximation that W is constant, the components of the states $\psi_\alpha{}^{(0)}$ may be obtained from Eq. (35):

$$\psi_{1\omega}{}^{(0)}=(1+s^2)^{-1/2}, \quad \psi_{2\omega}{}^{(0)}=s(1+s^2)^{-1/2},$$
$$\psi_{1\phi}{}^{(0)}=-s(1+s^2)^{-1/2}, \quad \psi_{2\phi}{}^{(0)}=(1+s^2)^{-1/2}. \tag{45}$$

From these, the transformed propagator Eq. (23) may be written down directly. Note that in this approximation of constant W, the remainder terms Q_{ij} in the propagator, vanish. We may also obtain the production and decay vertices for the particles $\omega^{(0)}$ and $\phi^{(0)}$ by applying this transformation to Eqs. (25a) and (25b):

$$V_\omega{}^{(0)}=(1+s^2)^{-1/2}[V_1+sV_2],$$
$$V_\phi{}^{(0)}=(1+s^2)^{-1/2}[-sV_1+V_2], \tag{46a}$$

and

$$D_\omega{}^{(0)}=(1+s^2)^{-1/2}[D_1+sD_2],$$
$$D_\phi{}^{(0)}=(1+s^2)^{-1/2}[-sD_1+D_2]. \tag{46b}$$

It is particularly convenient to consider those special production or decay states for which the vanishing of some of the vertex functions D or V is guaranteed by selection rules. For example, consider the case in which $K\bar{K}$ pairs are observed as the decay product. Then, since the unitary singlet does not interact with the $K\bar{K}$ system,

$$D_1(K\bar{K})=0.$$

Therefore Eq. (46b) can be simplified to read

$$D_\omega{}^{(0)}(K\bar{K})=s(1+s^2)^{-1/2}D_2(K\bar{K})=sD_\phi{}^{(0)}(K\bar{K}). \tag{47}$$

Other similar relationships may be found in cases where analogous selection rules apply.

VII. ELECTROMAGNETIC MIXING

We now consider the full matrix W given by Eq. (28). The first step is to write W in the new representation which diagonalizes w, namely,

$$W'(z)=\Omega^{(0)}(z)W(z)\Omega^{(0)}(z). \tag{48}$$

In the approximation that the z dependence of W may be ignored, the matrix $\Omega^{(0)}$ is found from Eq. (35) by setting $s_\phi=s_\omega=s$. Then

$$W'=\begin{bmatrix} z_\omega{}^{(0)} & 0 & q \\ 0 & z_\phi{}^{(0)} & q' \\ q & q' & z_\rho{}^{(0)} \end{bmatrix} \tag{49}$$

with

$$q=(e+e's)(1+s^2)^{-1/2},$$
$$q'=(-se+e')(1+s^2)^{-1/2}, \tag{50}$$

and

$$z_\rho{}^{(0)}=m_3{}^2,$$

the last notation being introduced merely for the sake of uniformity.

Although the electromagnetic coupling measured by q and q' is presumably small, the perturbation q will have a large effect because the ρ-ω mass difference is comparable to their widths. On the other hand, q' may be treated as a small perturbation since the ϕ mass is far removed from the other two. The effect of this perturbation is not large enough to be interesting at present. Therefore, we need only to consider the diagonalization of the submatrix

$$v'=\begin{pmatrix} z_\omega{}^{(0)} & q \\ q & z_\rho{}^{(0)} \end{pmatrix} \tag{51}$$

of W'. Here v' may be treated as a constant independent of z.

The solutions, z_α, of the equation

$$\det[v'-z_\alpha]=0 \tag{52}$$

are then, to a good approximation, the desired poles. In analogy to Eqs. (38), they are found to be

$$z_\omega=z_\omega{}^{(0)}+q\sigma,$$
$$z_\rho=z_\rho{}^{(0)}-q\sigma, \tag{53}$$

where, as in Eq. (39),

$$\sigma^2=(z_\rho-z_\rho{}^{(0)})/(z_\rho-z_\omega{}^{(0)}). \tag{54}$$

Furthermore, the analog of Eq. (35) is

$$\psi_\omega=(1+\sigma^2)^{-1/2}[\psi_\omega{}^{(0)}+\sigma\psi_\rho{}^{(0)}],$$
$$\psi_\rho=(1+\sigma^2)^{-1/2}[-\sigma\psi_\omega{}^{(0)}+\psi_\rho{}^{(0)}], \tag{55}$$

where $\psi_\rho{}^{(0)}\equiv|3\rangle$.

The parameter σ is sensitive to small changes in the "zero-order" masses $z_\omega{}^{(0)}$ and $z_\rho{}^{(0)}$ because their difference is comparable to their imaginary parts. Therefore σ must be treated as an unknown complex mixing parameter. The location of the poles z_ω and z_ρ may be determined from the experimental data by examining in detail the energy dependence of the 2π and 3π production amplitudes. To describe the production amplitudes, note that, according to Eqs. (55) and (45) the components of the ψ_α are:

$$\psi_{1\omega}=(1+\sigma^2)^{-1/2}(1+s^2)^{-1/2},$$
$$\psi_{2\omega}=s(1+\sigma^2)^{-1/2}(1+s^2)^{-1/2},$$
$$\psi_{3\omega}=\sigma(1+\sigma^2)^{-1/2},$$
$$\psi_{1\rho}=-\sigma(1+\sigma^2)^{-1/2}(1+s^2)^{-1/2}, \tag{56}$$
$$\psi_{2\rho}=-\sigma s(1+\sigma^2)^{-1/2}(1+s^2)^{-1/2},$$
$$\psi_{3\rho}=(1+\sigma^2)^{-1/2},$$

and the components of ψ_ϕ are the same as those of $\psi_\phi{}^{(0)}$. These components may be inserted into Eq. (23) to obtain the complete expression for the transformed propagator. The production and decay vertices are found from Eqs. (25a) and (25b):

$$V_\omega = (1+\sigma^2)^{-1/2}[V_\omega{}^{(0)}+\sigma V_3],$$
$$V_\rho = (1+\sigma^2)^{-1/2}[-\sigma V_\omega{}^{(0)}+V_3], \quad (57a)$$
$$V_\phi = V_\phi{}^{(0)},$$

and

$$D_\omega = (1+\sigma^2)^{-1/2}[D_\omega{}^{(0)}+\sigma D_3],$$
$$D_\rho = (1+\sigma^2)^{-1/2}[-\sigma D_\omega{}^{(0)}+D_3], \quad (57b)$$
$$D_\phi = D_\phi{}^{(0)},$$

where $V_\omega{}^{(0)}$, $V_\phi{}^{(0)}$, $D_\omega{}^{(0)}$, and $D_\phi{}^{(0)}$ are given by Eqs. (46).

VIII. SOME APPLICATIONS TO PRODUCTION AMPLITUDES

The production amplitudes are given by Eq. (24) in terms of the V_α and D_α which, in turn, may be obtained from Eqs. (57) and (46). We shall focus our attention on the behavior of the amplitudes near one of the resonances so that the slowly varying remainder term in Eq. (24) will be dropped. In this approximation, the production amplitude takes the form

$$A_{dp}(k^2) = \frac{D_\omega(d)V_\omega(p)}{k^2-z_\omega} + \frac{D_\rho(d)V_\rho(p)}{k^2-z_\rho} + \frac{D_\phi(d)V_\phi(p)}{k^2-z_\phi}, \quad (58)$$

where $z_\phi = z_\phi{}^{(0)}$.

When the 2π and 3π decay modes of the ω and ρ are considered, the expressions Eq. (57) are simplified by the fact that the states $\psi_\omega{}^{(0)}$, $\psi_\phi{}^{(0)}$ cannot decay into the 2π state and the state $|3\rangle \equiv \psi_\rho{}^{(0)}$ cannot decay into the 3π state. Therefore

$$D_\omega(2\pi) = \sigma D_\rho(2\pi),$$
$$D_\rho(3\pi) = -\sigma D_\omega(3\pi), \quad (59)$$
$$D_\phi(2\pi) = 0.$$

In the immediate neighborhood of the ρ and ω poles, the contribution of the ϕ pole may be treated as a constant which is ignored along with other nonresonant contributions that have been dropped. Then the amplitudes for production of the 2π and 3π modes given by Eq. (58) are

$$A_{2\pi}(k^2) = \sigma D_\rho(2\pi)V_\omega\left[\frac{\zeta}{k^2-z_\rho}+\frac{1}{k^2-z_\omega}\right], \quad (60a)$$

$$A_{3\pi}(k^2) = D_\omega(3\pi)V_\omega\left[\frac{-\sigma^2\zeta}{k^2-z_\rho}+\frac{1}{k^2-z_\omega}\right], \quad (60b)$$

where

$$\zeta = V_\rho/\sigma V_\omega. \quad (61)$$

Equations (60) can then be further simplified to

$$A_{2\pi} \propto \frac{k^2-m_d{}^2+im_d\Gamma_d}{[k^2-z_\rho][k^2-z_\omega]}, \quad (62a)$$

$$A_{3\pi} \propto \frac{k^2-m_d'^2+im_d'\Gamma_d'}{[k^2-z_\rho][k^2-z_\omega]}. \quad (62b)$$

The form of these amplitudes is particularly simple. Each is the product of two Breit-Wigner resonance curves in the square of the energy times an inverse Breit-Wigner curve or a "dip." The location and width of the two resonant peaks are independent of the mixing parameter σ and the production process and are given simply by z_ω and z_ρ. The location, m_d, and width, Γ_d, of the dip depend on the production process and the mixing parameter as well as the masses and lifetimes.

Without attempting to fit data, certain qualitative remarks can be made. If one pion exchange processes dominate in the reaction $\pi^- + p \to 3\pi + n$, then it follows that the corresponding production vertex $V_\omega{}^{(0)}=0$ and hence, according to Eqs. (57a) and (61), $\zeta = 1/\sigma^2$. Therefore the coefficients of the first and second terms in Eq. (60b) are equal and opposite with the result that the numerator in Eq. (62b) is constant. The shape of that spectrum will then be dominated by the resonance curve of smaller width.

Similarly if one studies $\pi^- + p \to 2\pi + n$ and selects only those events proceeding through one-ρ exchange processes as shown in Fig. 2, then $V_3=0$ and $\zeta=-1$. Hence the two terms in Eq. (60a) have equal and opposite coefficients, and the spectrum is again dominated by the narrower resonance. Since the ω width is considerably smaller than the ρ width, the term $1/[k^2-z_\omega]$ will dominate and the 2π mass spectrum will peak at $k^2 = m_\omega{}^2$.

It is obvious from the above that the position of the peaks and the shape of the curve depend on the production mechanism, and hence on the momentum transfer. Therefore a variation in the apparent location of resonance with momentum transfer is to be expected as noted by Bernstein and Feinberg.[2]

It is interesting to note that even if the ω and ρ masses are nearly degenerate, a mass splitting can be induced by the interference dip to give masses in agreement with experiment. Thus the curves shown in Figs. 3 and 4 of the 2π and 3π mass spectrum, assuming on the one hand (Fig. 3) a near degeneracy of the ρ and ω masses and on the other hand (Fig. 4) that z_ρ and z_ω are close to the experimentally observed peaks, bear a close resemblance to one another.

However the 2π amplitude drops off more slowly outside the resonance region in the degenerate case, which leads to an apparent broadening of the peaks. Since the interference of the resonances with background effects has not been taken into account, these details are not to be taken too seriously and the result is to be looked upon as the suggestion of a qualitative effect.

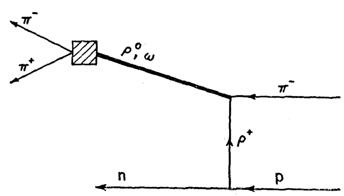

FIG. 2. Diagram illustrating 2π production in π-p interaction proceeding via one-rho exchange.

Equation (58) may also be applied to the study of the relationship between ω and ϕ production processes. In this case, attention is directed to resonance states with $G=-1$; hence, the contributions of the ρ pole term in Eq. (58) may be dropped and the relevant production amplitude is

$$A_{dp}(k^2)=\frac{D_\omega(d)V_\omega(p)}{k^2-z_\omega}+\frac{D_\phi(d)V_\phi(p)}{k^2-z_\phi}. \qquad (63)$$

It is known[9] that the $\rho\pi$ decay rate of the physical ϕ state is strongly inhibited, that is, the ratio of $\rho\pi$ to $K\bar{K}$ intensities in an arbitrary production process appears to

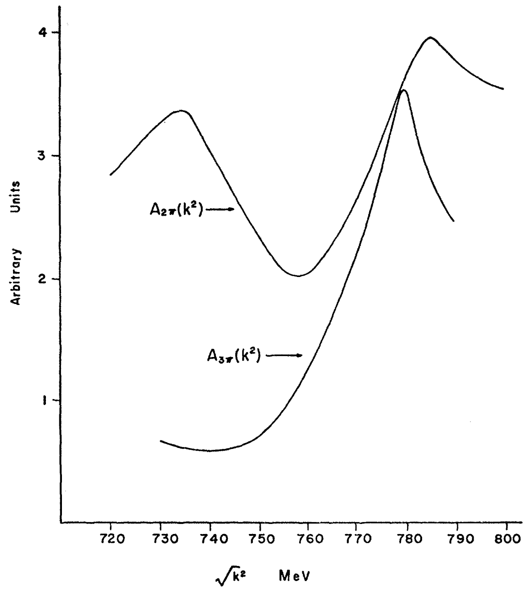

FIG. 3. Production amplitudes for 2π and 3π with nearly degenerate ρ, ω masses, calculated from Eq. (62) using: $m_\rho=755$ MeV, $\Gamma_\rho=90$ MeV, $m_\omega=765$ MeV, $\Gamma_\omega=60$ MeV, $m_d=760$ MeV, $\Gamma_d=30$ MeV, $m_d'=745$ MeV, $\Gamma_d'=30$ MeV, and a common value of s, V_ω, and V_ρ.

be very small near $k^2=z_\phi$. This ratio is proportional to

$$\left[\frac{|A_{\rho\pi}|^2}{|A_{K\bar{K}}|^2}\right]_{k^2=z_\phi}=\frac{|D_\phi(\rho\pi)|^2}{|D_\phi(K\bar{K})|^2}. \qquad (64)$$

The value of $D_\phi(K\bar{K})$ is given by Eqs. (57b) and (47) while $D_\phi(\rho\pi)$ is given by Eqs. (57b) and (46). From the experimental evidence that $D_\phi(\rho\pi)$ is very small it may be concluded from Eq. (46) that

$$s\approx D_2(\rho\pi)/D_1(\rho\pi). \qquad (65)$$

This result appears to involve *two* accidental equalities, one for the real and the other for the imaginary parts of the expressions on the two sides of the equations. However, since $\mathrm{Im}(s)$ appears to be very small, as indicated in Eq. (43), the condition

$$\mathrm{Re}s=\mathrm{Re}[D_2(\rho\pi)/D_1(\rho\pi)] \qquad (66)$$

[9] P. L. Connolly, E. L. Hart, K. W. Lai, G. London, G. C. Moneti *et al.*, Phys. Rev. Letters **10**, 371 (1963).

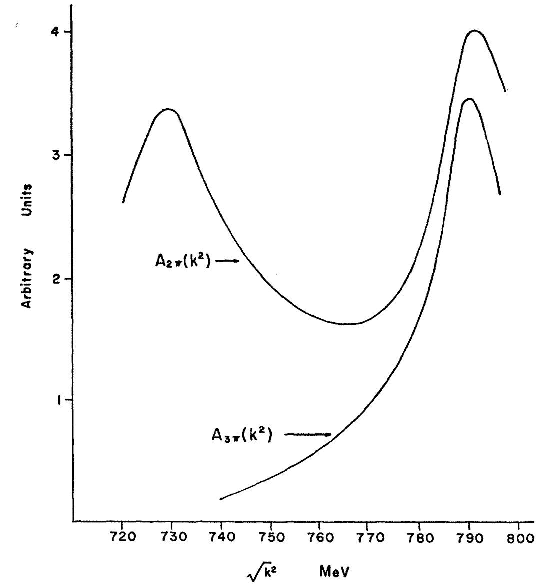

FIG. 4. Production amplitudes for 2π and 3π assuming nondegenerate ρ, ω masses, calculated from Eq. (62) using: $m_\rho=730$ MeV, $\Gamma_\rho=50$ MeV, $m_\omega=790$ MeV, $\Gamma_\omega=20$ MeV, $m_d=760$ MeV, $\Gamma_d=60$ MeV, $m_d'=705$ MeV, $\Gamma_d'=40$ MeV, and a common value of s, V_ω, and V_ρ.

may account for the experimental result if $\mathrm{Im}D_2(\rho\pi)/D_1(\rho\pi)$ is also small. In view of the contribution of the imaginary parts, the existence of a weak $\rho\pi$ mode of decay of the ϕ particle would be expected.

APPENDIX

The formal expression Eq. (14) for the propagator matrix,

$$\langle i|\Delta_F'(x'-x)|j\rangle=\langle 0|\{\varphi_i(x')\varphi_j(x)\}_T|0\rangle, \qquad (A1)$$

requires a generalization of the usual definition of the Heisenberg fields $\varphi_i(x)$ since the character of the bare particle "i" changes in time due to particle mixing. In principle, a more direct procedure would be to define the propagator in the physical particle representation, denoted by labels α, in which there is no mixing. However it is just the transformation to these states which we seek and we wish to express it in terms of the states $|j\rangle$ for which the quantum numbers, such as SU_3 representation and isotopic spin, are well defined. It is possible to define a field $\varphi_j^{\mathrm{in}}(x)$ that develops in time like a free field, and therefore may be used to define states having the quantum numbers j. Then the Heisenberg fields labeled with j may be defined as those fields $\varphi_j(x)$ which satisfy the equations of motion governed by the full Hamiltonian and the boundary conditions

$$[\varphi_j(x)]_{t\to-\infty}=[\varphi_j^{\mathrm{in}}(x)]_{t\to-\infty} \qquad (A2)$$

in the usual sense of Lehmann, Symanzik, and Zimmermann. The connection between the fields may then be expressed by the formal equation

$$\varphi_j(x)=\varphi_j^t(x)=U(t,-\infty)\varphi_j^{\mathrm{in}}(x)U(-\infty,t), \qquad (A3)$$

where $\varphi_j^{\tau}(x)$ is the free field satisfying the condition $[\varphi_j^{\tau}(x)]_{t=\tau}=[\varphi_j(x)]_{t=\tau}$ and the unitary transformation $U(t,t')$ is given by

$$U(t,t')=\sum_{n=0}^{\infty}\frac{i^n}{n}\int_t^{t'}d^4x_1\cdots\int_t^{t'}d^4x_n \times\{L^{(t)}(x_1)\cdots L^{(t)}(x_n)\}_T. \quad \text{(A4)}$$

Here, $L^{(\tau)}(x)$ is the interaction Lagrangian in interaction representation, which may be obtained by replacing $\varphi_j(x)$ by $\varphi_j^{\tau}(x)$ in the interaction $L(x)$. We note that

$$U(t,t')U(t',t'')=U(t,t''), \quad \text{(A5)}$$

whence

$$U^{-1}(t,t')=U(t',t). \quad \text{(A6)}$$

In order to establish that the propagator is given by Eq. (A1), we introduce a fictitious independent field $f_j(x)$, characterized by the same quantum numbers j, which has a very weak interaction of the form

$$L'(x)=\epsilon\sum_j f_j(x)\varphi_j(x) \quad \text{(A7)}$$

with the field $\varphi_j(x)$ and no other interaction. Then, since the only mixing of the fictitious particle states is due to mixing in $\varphi_j(x)$, the propagator matrix $\Delta_F'(x-y)$ for the fields φ_j may be determined by considering the self-energy diagrams for the fictitious particles to order ϵ^2. Since ϵ may be made as small as we please, orders higher than ϵ^2 may be ignored. Therefore we require the element of the S matrix

$$S=\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty}d^4x_1\cdots\int_{-\infty}^{\infty}d^4x_n\{I^{\rm in}(x_1)\cdots I^{\rm in}(x_n)\}_T \quad \text{(A8)}$$

between states $|i\rangle$ and $|j\rangle$ of the fictitious particle. Here

$$I(x)=L(x)+L'(x)$$

is the total interaction. Only the term $S^{(2)}$ of order ϵ^2 is of interest here:

$$S^{(2)}=\sum_{n=2}^{\infty}\frac{i^n}{n!}\frac{n(n-1)}{2}\int_{-\infty}^{\infty}d^4x_1\cdots\int_{-\infty}^{\infty}d^4x_n \times\{L^{\rm in}(x_1)\cdots L^{\rm in}(x_{n-2})L'^{\rm in}(x_{n-1})L'^{\rm in}(x_n)\}_T. \quad \text{(A9)}$$

Since now only the factors involving x_{n-1} and x_n refer to the fictitious particles we set $x_n=x$ and $x_{n-1}=x'$, make use of the fact that the fields $f_j(x)$ commute with $\varphi_k(x)$, and write

$$S^{(2)}=-\frac{\epsilon^2}{2}\sum_{j,k}\int_{-\infty}^{\infty}d^4x' f_j^{\rm in}(x') \times\int_{-\infty}^{\infty}d^4x f_k^{\rm in}(x)\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty}d^4x_1\cdots\int_{-\infty}^{\infty}d^4x_n \times\{L^{\rm in}(x_1)\cdots L^{\rm in}(x_n)\varphi_j^{\rm in}(x')\varphi_k^{\rm in}(x)\}_T. \quad \text{(A10)}$$

From this form of the matrix element it is evident that the matrix element of the propagator is

$$\langle j|\Delta_F'(x'-x)|k\rangle=\left\langle 0\left|\sum_{n=0}^{\infty}\frac{i^n}{n!}\int_{-\infty}^{\infty}d^4x_1\cdots\int_{-\infty}^{\infty}d^4x_n \times\{L^{\rm in}(x_1)\cdots L^{\rm in}(x_n)\varphi_j^{\rm in}(x')\varphi_k^{\rm in}(x)\}_T\right|0\right\rangle. \quad \text{(A11)}$$

The range of integration for each x_i may be divided into the interval between t' and t, the region above this interval, and the region below it. When $t'>t$ we then have

$$\langle j|\Delta_F'(x'-x)|k\rangle=\langle 0|\sum_{n=0}^{\infty}\frac{i^n}{n!}\sum_{\lambda,\mu,\nu}^{\infty}\frac{n!}{\lambda!\mu!\nu!} \times\int_{t'}^{\infty}d^4x_1\cdots\int_{t'}^{\infty}d^4x_\lambda\{L^{\rm in}(x_1)\cdots L^{\rm in}(x_\lambda)\}_T \times\varphi_j^{\rm in}(x')\int_t^{t'}d^4x_1\cdots\int_t^{t'}d^4x_\mu\{L^{\rm in}(x_1)\cdots L^{\rm in}(x_\mu)\}_T \times\varphi_k^{\rm in}(x)\int_{-\infty}^{t}d^4x_1\cdots\int_{-\infty}^{t}d^4x_\nu \times\{L^{\rm in}(x_1)\cdots L^{\rm in}(x_\nu)\}_T, \quad \text{(A12)}$$

where $\lambda+\mu+\nu=n$. According to Eqs. (A3), (A4), and (A6),

$$\langle j|\Delta_F'(x'-x)|k\rangle =\langle 0|U(-\infty,t')U(t',\infty)U(t',-\infty)\varphi_j^{\rm in}(x')U(-\infty,t) \times U(t',t)U(t,-\infty)\varphi_k^{\rm in}(x)U(t,-\infty)|0\rangle. \quad \text{(A13)}$$

From Eq. (A5) it follows that

$$U(-\infty,t')U(t',\infty)=S^{(0)}, \quad \text{(A14)}$$

the operator S for $\epsilon=0$, and Eq. (A3) then yields

$$\langle j|\Delta_F'(x'-x)|k\rangle=\langle 0|S^{(0)}\varphi_j(x')\varphi_k(x)|0\rangle \quad \text{(A15)}$$

in the case $t'>t$. When $t'<t$, the same procedure yields a similar result with $\varphi_j(x')$ and $\varphi_k(x)$ interchanged. When these results are combined with the fact that $S^{(0)}|0\rangle=|0\rangle$, the suggested Eq. (A1) is obtained.

It should be emphasized that, in addition to the usual limitations of such a formal argument, this treatment involves the further assumption that the field operators describe stable particles, so that asymptotic fields may be given a reasonable definition. Since the reciprocal lifetime is fairly small compared to the mass of the particle for each one in question, this assumption may not be too unreasonable.